Journal of Engineering Physics and Thermophysics, Vol. 71, No. 1, 1998

A METHOD OF RAPID IDENTIFICATION OF HEAT FLUXES

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UDC 536.24

In design, development, and operation of crucial engineering systemssubjected to high heat fluxes, it is often necessary to observe the thermal state of the object in real time. The authors suggest an approach that is based on the methodology of solution of inverse heat-conduction problems (IHCP) and is a special adaptation of these methods for solution of observation problems. The approach is based on the idea of the possibility of identifying the current thermal state of the object with the use of measurements that are chronologically close to the current time, which leads to formulation of the retrospective boundary-value IHCP in a local time interval. A solution to this problem is considered, and results and estimates of the accuracy of simulation are presented.

Introduction. The functioning of many modern engineering systems is accompanied by high-intensity heat transfer processes that can be caused by interaction with the environment and by operation of power plants. Therefore, optimization of heat regimes is an important component of systems subjected to high heat fluxes. The most general trends consist in simulation of heat transfer processes in members of the structure subjected to high heat fluxes with subsequent choice of their operating characteristics so that their serviceability be ensured with a certain safety factor. However, situations are quite possible in which such an approach leads to the choice of nonoptimum designs. This can be explained by the complexity of processes considered and, as a result, by incomplete adequacy of the mathematical models that are used for their description and by the effect of various random factors an account of which is often absolutely impossible. In particular, the matter of optimality becomes especially important in development of reusable space systems and long-service apparatus, in which excess of the safety factor has an important effect on the final efficiency and cost. Therefore, in some cases it would be reasonable to use intelligent heat systems, i.e., systems in which control by feedback could be implemented, which is fully consistent with advanced trends in technology.

We can give some examples of various versions of active thermal-protection systems such as convective, film, and porous cooling of structures under high heat fluxes. In these systems, the pumping intensity of the heat-transfer agent or the mass flow rate of the coolant can be used as the controlling parameter, and the heat flux into the heated wall or its temperature, as the controlled one. Simultaneously, these quantities are observed characteristics. In most cases direct measurement of heat fluxes or temperatures of heated surfaces is difficult; however, one can obtain results by solving the corresponding IHCP on the basis of readings of thermal sensors that are mounted inside the wall or on its internal boundary. It is evident that such adaptive systems could allow optimization of the power of the coolant pumping systems or the coolant flow rate.

Thermal experiments and tests are another area where control of heat fluxes based on the servo principle or continuous diagnostics of the temperatures of the objects may be required. In the first case, it is necessary to reproduce specified time dependences of heat fluxes or temperatures in specimens or models on the basis of information obtained from control sensors, while in the second case it is necessary to process experimental data immediately in order to determine the time of emergency stop of the tests to avoid damage to expensive equipment.

To develop thermal systems in which control by feedback is performed, it is necessary first of all to solve the problem of observation of the thermal state of the object in real time. (The thermal state is as the temperature

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Moscow Aircraft Institute, Moscow, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 71, No. 1, pp. 30-40, January-February, 1998. Original article submitted October 22, 1997.

1062-0125/98/7101-0025$20.00

distribution over the coordinates and the conditions of heat transfer at the boundaries of the object.) In this case use of the methodology of the IHCP brings about important new capabilities. However, two problems are involved here. The first is associated with the physical nature of the heat-conduction process and is expressed in delay of the response to the unknown action and its substantial damping. On the one hand, this is a physical reason for the incorrectness of the IHCP and all resultant mathematical properties. On the other (which is also very important in the present case), any estimate of the current thermal state of the object is only an extrapolation. As is shown by calculations, the extrapolation error can be admissible high and estimates of reasonable accuracy can be obtained only retrospectively.

The second problem is construction of algorithms that are adapted in a special way to rapid solution of the IHCP. In this case it is desirable that the time of solution of the problem be a small part of the interval of the forced retrospective shift. The combination of fast algorithms with the capabilities of modern special computers based on of the third generation of signal processors inspires hope for a successful solution to this problem.

Successive algorithms in which a limited sample is used for estimation of the current thermal state of the object are the most suitable for solution of the IHCP in real (or close to real) time. In considering the available methods, it is necessary first of all to pay attention to the method of construction of the the successive procedure and the means of inclusion of the nonlinearity and incorrectness of the problem. The following methods can be used here: direct numerical methods based on the finite-difference representation of the heat-conduction equation [1-3], the method of optimum dynamic filtration [4], and methods of successive functional approximation [2]. All these methods have some drawbacks, among which the most important is that these algorithms are constructed so that a subsequent estimate of the thermal state of the object is computed on the basis of the previous one. First, this means that it is necessary either to have an estimate of the thermal state of the object at some time, which is assumed to be the initial time, or to develop the transient regime and be sure that the estimation process converges. In practice, this necessitates, for example, inclusion of algorithms before the start of the thermal process and their subsequent continuous operation. Second, with this method of solution, computational errors in the current time cross section are transferred to the subsequent ones and it cannot be excluded that under certain conditions they will accumulate.

In what follows, a method is suggested for construction of successive algorithms that is based on consideration of the studied thermal process within a certain local interval that precedes the current time. In this case, along with the boundary condition, the initial temperature distribution is also unknown. In this sense, according to the classification of [1], this formulation of the IHCP can be classified as a retrospective boundary-value problem. This formulation is sustainable in the sense that for solution of the problem, no initial information about the previous states is required (although if this information is available, it can be used). Successive repetition of this procedure with a chosen time step allows recovery of the thermal state of the object in a time segment of arbitrary length. Since estimation events in successive local intervals are independent, in principle the length of the time shift between them can be any value, in particular, longer than the interval itself, and it is determined by the expenditure on estimation in the local interval. In other words, in any case, estimates of the thermal state of the object can be obtained, and it is only necessary to determine what the value of the time step is and whether the latter is sufficient for subsequent control.

We consider the inverse problem for the quasilinear heat-conduction equation (the coefficients of the equation depend on the temperature). However, in the first part we investigate the solution of the retrospective boundary-value IHCP for the linear heat-conduction equation with constant coefficients, for which we obtain a regularized inverse operator that depends explicitly on the regularization parameter. In doing this, to choose optimum values for the local interval, the retrospective shift, and the step between measurements, use will be made of an approach based on analysis of the accuracy of the solution. In the second part, we consider the solution of the IHCP for the quasilinear equation. In essence, we use a regularized variant of the method of successive approximations in which in each iteration a linear IHCP that will be considered in detail later is solved. The conditions for conversion of this iteration process are analyzed. A method for choosing the regularization parameter that would ensure the most rapid convergence and a stopping rule are suggested. Accuracy estimates are presented.

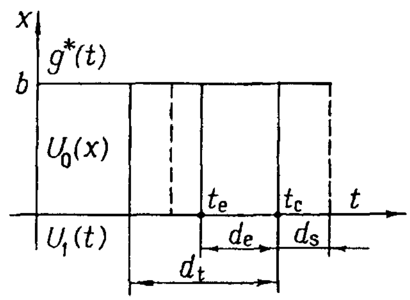

Fig. 1. Scheme of the time intervals.

1. Solution of the Linear IHCP We consider the scheme of the problem shown in Fig. 1. Let t_c be the current time. An interval that precedes t_c and whose length d_t is assumed to be short compared to the total time of observation of the heat-conduction process is called an estimation interval. At $t_0 = t_c - d_t$, the temperature distribution is assumed unknown and we determine it simultaneously with the sought boundary condition in the interval. Now, we choose a certain point t_e in the estimation interval, and the value of the boundary condition at that time is assumed to be the sought point estimate. The difference $d_e = t_c - t_e$ is called the time delay of the estimate relative to the current time. As the current time moves along the time axis, a sequence of point estimates is obtained and these estimates discretely reconstruct the sought boundary condition.

In the estimation interval, the new time variable $\tau = t - (t_c - d_t)$ is introduced and the inverse problem is written in dimensionless form:

$$T_\tau = T_{xx}, \quad x \in [0, 1], \quad \tau \in [0, d_t); \quad T(x, 0) = U_0(x), \quad x \in (0, 1);$$
$$\mu T(0, \tau) + \nu\lambda T_x(0, \tau) = U_1(\tau), \quad \tau \in (0, d_t);$$
$$\lambda T_x(1, \tau) = g^*(\tau), \quad \tau \in (0, d_t); \tag{1.1}$$
$$T(x_k, \tau) + \xi_k(\tau) = f_k^*(\tau), \quad k = \overline{1, N}, \quad \tau \in (0, d_t).$$

where $T(x, \tau)$ is the temperature; μ, ν are parameters equal to 0 or 1; x_k are the coordinates of the temperature sensors; $\xi_k(\tau)$ is the measurement error of the sensors; $k = \overline{1, N}$ is the number of the sensor. It is necessary to determine the functions $U_0(x)$ and $U_1(\tau)$ using measurements of $f_k^*(\tau)$, $k = \overline{1, N}$, and $g^*(\tau)$. In the operator form problem (1.1) has the form $AU = f$, where A is the operator generated by the heat-conduction equation, f is a function of measured quantities, and $U = \{U_0(x), U_1(\tau)\}$.

In the case $x_N = 1$, with some additional assumptions that are not too restrictive from the point of view of practice, uniqueness of the solution of inverse problem (1.1) follows from [5], where uniqueness of the solution of the Cauchy problem was proved for the equation

$$a(\tau, x) T_\tau + b(\tau, x) T_x + c(\tau, x) T = T_{xx} \tag{1.2}$$

in a rectangular region with the following limitations on the coefficients of Eq. (1.2):

$$0 < a(\tau, x) \le 1, \quad |a_\tau(\tau, x)| < 1, \quad 0 < c(\tau, x) \le 1, \quad |b(\tau, x)| \le 1,$$
$$|b_\tau(\tau, x)| < 1. \tag{1.3}$$

In this case with a slight modification of the technique of the proof, it is possible to extend the results to the case $|c(\tau, x)| < 1$.

It is also known that inverse problem (1.1) in ill-posed as regards violation of the stability condition. Because of this, special regularizing algorithms are needed to solve it [6]. In [1] it is shown that the assumption that the sought function belongs to the Sobolev space W_2^2 is efficacious in solving the IHCP. We make this assumption, and using A. N. Tikhonov's regularization method, we determine the solution of the IHCP considered from the condition of a minimum for the stabilizing functional

$$J_\alpha = \| A U - f\|^2_{L_2} + \alpha \left(\| U_0 (x) \|^2_{W_2^2} + \| U_1 (\tau) \|^2_{W_2^2} \right), \quad \alpha \ge 0 . \tag{1.4}$$

We approximate the sought relations over some systems of the basis functions $\{\varphi_j(x)\}_1^{n1}$, $\{\psi_j(\tau)\}_1^{n2}$ in the form

$$U_0 (x) = \sum_{j=1}^{n1} \beta_j \varphi_j (x), \quad x \in (0, 1), \tag{1.5}$$

$$U_1 (\tau) = \sum_{j=1}^{n2} \gamma_j \psi_j (\tau), \quad \tau \in (0, d_t), \tag{1.6}$$

divide the estimation interval into $m-1$ equal parts, and take m discrete readings of the observed functions, which will result in a finite-dimensional formulation of problem (1.1). With account for (1.5) and (1.6), the initial operator equation can be written in the form

$$A\bar{p} - \bar{z} = 0, \tag{1.7}$$

where $\bar{p} = [\bar{\beta}, \bar{\gamma}]^T$; $z_{i\times k} = f_k^*(\tau_i) - T(x_k, \tau_i, g^*(\tau))$, $i = \overline{1, m}$, $k = \overline{1, N}$, and functional (1.4) can be written as

$$J_\alpha = \| A_1 \bar{\beta} + A_2 \bar{\gamma} - \bar{z} \|^2_{R^{mN}} + \alpha \left(\| F_1 \bar{\beta} \|^2_{R^{n1}} + \| F_2 \bar{\gamma} \|^2_{R^{n2}} \right) =$$

$$= \| A \bar{p} - \bar{z} \|^2_{R^{mN}} + \alpha \| F \bar{p} \|^2_{R^n} . \tag{1.8}$$

Here F is a block-diagonal matrix whose elements F_1 and F_2 are determined from the relations

$$F_1^T F_1 = \Phi_1, \quad \text{where} \quad \Phi_1 (i, j) = \langle \varphi_i (x), \varphi_j (x) \rangle_{W_2^2}, \quad i = \overline{1, n1}, \quad j = \overline{1, n1},$$

$$F_2^T F_2 = \Phi_2, \quad \text{where} \quad \Phi_2 (i, j) = \langle \psi_i (\tau), \psi_j (\tau) \rangle_{W_2^2}, \quad i = \overline{1, n2}, \quad j = \overline{1, n2},$$

$T(x_k, \tau_i, g^*(\tau))$ is the solution of system (1.1) under the condition $U_0(x) \equiv 0$, $U_1(\tau) \equiv 0$.

If $\tilde{A} = AF^{-1}$ and $\tilde{A} = USV^T$ is the result of singular expansion of $\tilde{A}$, then following [7], the solution of the least-squares problem that minimizes functional (1.8) can be written in the form

$$\hat{\bar{p}} = B (\alpha) \bar{z} = F^{-1} V S_\alpha U^T \bar{z} = F^{-1} V S_\alpha \bar{g}_1 . \tag{1.9}$$

Here $\bar{g}_1$ is the first n elements of the vector $\bar{g} = U^T \bar{z}$, S_α is a diagonal matrix whose elements are equal to

$$s_{\alpha i} = s_i / (s_i^2 + \alpha), \tag{1.10}$$

where s_i are singular numbers of the matrix $\tilde{A}$. Factorization of $\tilde{A}$ is carried out preliminarily, and therefore solution of the problem is reduced to a sequence of multiplications of matrices by the vector $\bar{z}$. On the order of $(mN)^2 + l_0 n + n^2$ operations are required to obtain an estimate. Here l_0 is the number of iterations in the search for the optimum value of the parameter α. Substitution of the estimate of $\bar{p}$ into (1.5) and (1.6) gives the sought boundary condition at any time in the interval d_t.

An important matter is the choice of the optimum regularization parameter α. With account for (1.7) and (1.9), the estimation error is specified by the expression

$$\bar{\varepsilon} (\alpha) = \hat{\bar{p}} - \bar{p} = (B (\alpha) A - I) \bar{p} + B (\alpha) \bar{\xi}, \tag{1.11}$$

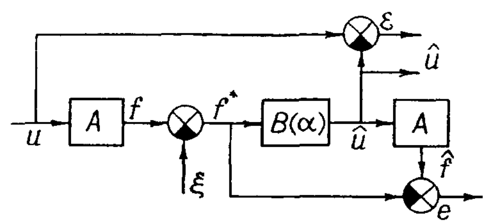

Fig. 2. Block diagram of the estimation system.

and the accuracy of the regularized solution is specified by the trace of the quadratic form $\Delta_L^2 = \mathrm{sp}(\mathrm{M}[\overline{\varepsilon}(\alpha)\overline{\varepsilon}^T(\alpha)])$. Let $\overline{e}(\alpha) = (I - AB(\alpha))\overline{f}^* = E(\alpha)\overline{f}^*$ be the residual of problem (1.7) and $\mathrm{M}[\overline{e}(\alpha)\overline{e}^T(\alpha)] = V_e$ be the covariance matrix of the residual. It is assumed that the measurement error is a steady-state random process with a zero mathematical expectation and a spesified covariance matrix $\mathrm{M}[\xi\,\xi^T] = V_\xi$. It is shown in [8] that the solution that is optimum in the sense of the minimum of Δ_L^2 occurs in the case where $\hat{V}_e = V_e = V_\xi E^T$. Here the left-hand side is the estimate of the covariance matrix of the residual that is obtained from the finite-volume sample. Therefore, we can speak only about adoption of the appropriate statistical hypothesis W_0, which can be checked with the statistic

$$r_w(\alpha) = \overline{e}^{\,T}(\alpha)\,(V_\xi E^T(\alpha))^{-1}\,\overline{e}(\alpha)\,, \quad \mathrm{M}\,[r_w(\alpha)] = mN\,, \tag{1.12}$$

If it is assumed that the matrix V_ξ is diagonal and all its elements are equal to σ^2 (i.e., the case of white noise is considered), statistic (1.12) can be evaluated from the formula

$$r_w(\alpha) = \frac{1}{\sigma^2}\,\overline{g}^{\,T}(I - SS_\alpha)\,\overline{g}\,. \tag{1.13}$$

Therefore, the above formula has a χ^2 distribution with mN degrees of freedom. Therefore, the hypothesis W_0 can be considered true if the value of statistic (1.12) belongs to the confidence interval

$$\Theta_{mN}(\gamma) = [\chi^2_{\gamma/2}(mN)\,,\ \chi^2_{1-\gamma/2}(mN)]\,, \tag{1.14}$$

where $\chi^2_{\gamma/2}$ is the quantile of the distribution χ^2 of the level $\gamma/2$. Values of the regularization parameter that specify the confidence region Ξ_Θ of variation of the parameter α correspond to boundaries of interval (1.14). For small m the confidence interval can be rather long.

The residual principle [9] is widely used for solution of ill-posed problems. Let $\overline{f}_k = T(x_k, \tau_i)$. Then, if $\|\overline{f} - \overline{f}^*\|_{R^{mN}} = \|\xi\|_{R^{mN}} \le \delta$ and $\|\overline{f}\|_{R^{mN}} \ge \delta$, then the root of the residual equation

$$\|A\hat{\overline{p}}(\alpha^*) - \overline{z}\|^2_{R^{mN}} = \delta^2\,. \tag{1.15}$$

should be taken as the optimum value α^*. No other assumptions relative to the measurement error ξ are made, besides the assumption of finiteness of its norm. If it is assumed that this is white noise of intensity σ, then to satisfy the condition of finiteness of the norm with a spesified probability of error of the first kind, we should take

$$\delta^2 = \chi^2_{1-\gamma/2}(mN)\,\sigma^2\,. \tag{1.16}$$

Use of (1.16) gives an estimate of the upper limit of the regularization parameter whose application leads to substantially smoothed results in most cases, which is especially pronounced for small m. It can be shown that $\overline{e}^T(\alpha)\overline{e}(\alpha)$ is always smaller than $r_w(\alpha)\sigma^2$, and in the choice of the regularization parameter according to (1.15), the statistic $r_w(\alpha)$ is always beyond the right boundary of the confidence interval. This means that such estimates are statistically inconsistent.

In order to choose a value of the parameter α within the confidence interval, we express our problem in the form of the linear system shown in Fig. 2. We consider linear operators $\varepsilon(\alpha) = H_1(\alpha)u$ and $\varepsilon(\alpha) = H_2(\alpha)\xi$ that relate the output ε to the inputs u and ξ. If a unit signal is input to the inputs, for the delay d_ε chosen, we obtain

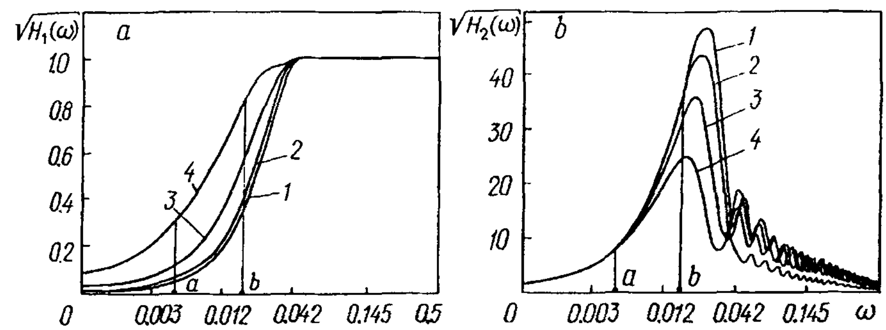

Fig. 3. Transfer functions: a) $\varepsilon(\omega) = H_1(\omega)u(\omega)$; b) $\varepsilon(\omega) = H_2(\omega)\xi(\omega)$; point a) 1 Hz; point b) $1/d_t$ Hz; $N = 1$, $x_1 = 1.0$, $d_t = 0.32$, $d_e = 0.16$, $m = 64$; 1) $\alpha = 0.0002$; 2) 0.002; 3) 0.02; 4) 0.2.

the discrete pulse characteristics $h_1^{(\alpha)}(t)$ and $h_2^{(\alpha)}(t)$ and, then, the transfer functions $H_1^{(\alpha)}(\omega)$ and $H_2^{(\alpha)}(\omega)$ (see, for example, Fig. 3). In this figure the abscissa is fractions of the frequency of the readings on a logarithmic scale, i.e., the true frequency in Hz is normalized on the frequency of readings $\omega_0 = m/d_t$ by division. It is assumed that the spectral density of the input signal u is the same and is nonzero only in the interval [0, 1) Hz (in dimensionless time), and the spectral density of noise is the same at all frequencies. With account for this, the amplification coefficients are calculated as

$$K_1(\alpha) = \int_0^{1/\omega_0} H_1^{(\alpha)}(\omega)\, d\omega\,, \quad K_2(\alpha) = \int_0^{0.5} H_2^{(\alpha)}(\omega)\, d\omega\,.$$

In this case we can write the expression for the relative error

$$\tilde{\varepsilon}(\alpha) = \sqrt{K_1^2(\alpha) + K_2^2(\alpha)\, k_1^2\, \tilde{\sigma}^2}\,, \tag{1.17}$$

where k_1 is a coefficient that allows for the time-average relation between the sought boundary condition and the maximum measured temperature, and $\tilde{\sigma}$ is a coefficient that specifies the measurement error in percent of the temperature. For example, we assume $k_1 = 0.5$ and $\tilde{\sigma} = 5\%$.

Functions $\tilde{\varepsilon}(\alpha)$ for different delays are given in Fig. 4 (the heat flux is being estimated). It can be seen that $\tilde{\varepsilon}(\alpha)$ has a region of minimum values, and for some d_e, there is a region of variation of α for which the relative error is almost the same. We specify a permissible level of the estimation error ε_{max} and isolate the region of variation of α for which $\tilde{\varepsilon} \le \varepsilon_{max}$. This region is referred to as the guaranteed-estimation interval and denoted by Ξ_ε. The average value of $\tilde{\varepsilon}(\alpha)$ in the interval Ξ_ε and the magnitude of Ξ_ε can be optimality criteria in the choice of the estimation interval d_t, the delay d_e, and the number of readings m. Thus, the following rule can be formulated for the choice of the regularization parameter α:

$$\forall \alpha \in \Xi = \Xi_\Theta \cap \Xi_\varepsilon\,. \tag{1.18}$$

When (1.18) is used, a statistically consistent solution is obtained that has satisfactory accuracy. If Ξ is an empty set, it is impossible to solve the problem in a particular time step with the available information.

From results of numerical experiments we have $d_t \in [0.25, 0.4]$ for the value of the estimation interval. The delay depends on the noise level of the measurements and the frequencies of the input action; however, with the assumptions for which the relative-error function shown in Fig. 4 was calculated, we may recommend $d_e > 0.12$ in recovery of the temperature and $d_e > 0.18$ in recovery of the heat flux; the number of readings in the estimation interval $m \approx 64$.

2. Solution of the Nonlinear IHCP. In the case where the thermophysical coefficients depend on the temperature, for an infinite plate of thickness b, the formulation of the IHCP has the form

$$C(T)\, T_\tau - (\lambda(T)\, T_x)_x = 0\,, \quad x \in [0, b]\,, \quad \tau \in [0, d_t)\,; \tag{2.1}$$

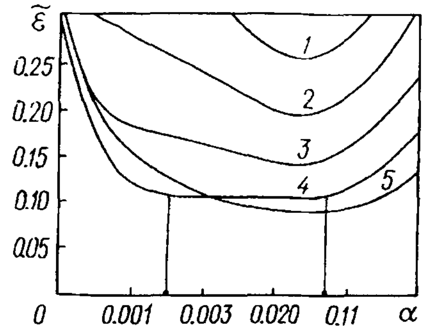

Fig. 4. Relative estimation error ($N = 1$, $x_1 = 1.0$, $d_t = 0.32$, $m = 64$); 1) $d_e =$ 0.09; 2) 0.12; 3) 0.15; 4) 0.18; 5) 0.21.

$$T(0, x) = U_0(x), \quad x \in (0, b); \tag{2.2}$$

$$\mu T(\tau, 0) + \nu\lambda(T(\tau, 0))\, T_x(\tau, 0) = U_1(\tau), \quad \tau \in (0, d_t); \tag{2.3}$$

$$-\lambda(T(\tau, b))\, T_x(\tau, b) = g^*(\tau), \quad \tau \in (0, d_t); \tag{2.4}$$

$$T(\tau, x_k) + \xi_k(\tau) = f_k^*(\tau), \quad k = \overline{1, N}, \quad \tau \in (0, d_t), \tag{2.5}$$

where $\lambda(T)$ and $C(T)$ are the thermal conductivity and the heat capacity, respectively. The boundary and initial conditions should be mutually consistent:

$$\mu U_0(0) + \nu\lambda(U_0(0))\, U_{0x}(0) = U_1(0), \quad -\lambda(U_0(b))\, U_{0x}(b) = g^*(0). \tag{2.6}$$

In solving of IHCP (2.1)-(2.5), as before, it is necessary to determine the functions $U_0(x)$ and $U_1(\tau)$ using measurements, according to (2.5).

We transform Eq. (2.1). Kirchhoff's substitution

$$\tilde{T} = P(T) = \int_0^T \lambda(T)\, dT,$$

results in transformation of initial equation (2.1) to the form

$$\tilde{C}(\tilde{T})\, \tilde{T}_\tau = \tilde{T}_{xx}, \tag{2.7}$$

where $\tilde{C}(\tilde{T}) = C(P^{-1}(\tilde{T})/\lambda(P^{-1}(\tilde{T}))$, and then we use the linear transformation $R(T) = 1 + \beta\tilde{T}$, where β is a factor chosen from the condition of the best approximation of the function $\tilde{C}(\tilde{T})$ by the function $R(\tilde{T})$. The equation in R also has the form of (2.7). It can easily be shown that after some simple transformations, the equation can be written as

$$R_\tau - \rho R_{xx} = -(\rho\tilde{C}(R) - 1)\, R_\tau. \tag{2.8}$$

We add the transformed initial and boundary conditions:

$$R(0, x) = 1 + \beta P(U_0(x)) \equiv \tilde{U}_0(x), \tag{2.9}$$

$$R(\tau, 0) = 1 + \beta P(U_1(\tau)) \equiv \tilde{U}_1(\tau), \tag{2.10}$$

$$-R_x(\tau, b) = \tilde{g}^*(\tau), \tag{2.11}$$

and the equation of observation

$$R_k^*(\tau) = R(\tau, x_k) + \zeta_k(\tau), \tag{2.12}$$

where $R(\tau, x_k) = 1 + \beta P(T(\tau, x_k))$, $\zeta_k(\tau) = \beta\lambda(T(\tau, x_k))\xi_k(\tau) \cong \beta\lambda(f_k^*(\tau))\xi_k(\tau)$.

In the case where one of the temperature sensors is located on the surface $x = b$, uniqueness of the solution of inverse problem (2.8)-(2.12) can be proved from results obtained in [5]. It is assumed that two solutions R_1 and R_2 exist that satisfy Cauchy conditions on the surface $x = b$. Then, the difference in these solutions ΔR satisfies an equation of the form

$$\tilde{a}(\tau, x)\,\Delta R_\tau + \tilde{c}(\tau, x)\,\Delta R = \Delta R_{xx},$$

where

$$\tilde{a}(\tau, x) = \tilde{C}(R_1(\tau, x))\,; \quad \tilde{c}(\tau, x) = \frac{\partial \tilde{C}(R_\xi)}{\partial R}(\tau, x)\frac{\partial R_2(\tau, x)}{\partial \tau}.$$

Here R_ξ is a certain solution chosen in accordance with the theorem of the mean. The time variable can always be scaled so that the coefficients $\tilde{a}(\tau, x)$ and $\tilde{c}(\tau, x)$ satisfy specified limitations on the absolute values. If in this case $|\tilde{a}_\tau(\tau, x)| < 1$, the solution to problem (2.8)-(2.12) is unique.

It is proved that to solve direct problem (2.8)-(2.11) we can use the iteration process

$$R_\tau^{l+1} - \rho R_{xx}^{l+1} = -(\rho\tilde{C}(R^l) - 1)\,R_\tau^l,$$

$$R^{l+1}(0, x) = \tilde{U}_0(x), \quad R^{l+1}(\tau, 0) = \tilde{U}_1(\tau), \quad R_x^{l+1}(\tau, b) = \tilde{g}^*(\tau),$$

which converges for any initial approximation if $0 < \rho < 2/\tilde{C}_1$ is chosen. In this case the convergence rate is determined by the quantity $\max(|1 - \rho\tilde{C}_0|, |1 - \rho\tilde{C}_1|)$. Here $\tilde{C}_0$ and $\tilde{C}_1$ are the upper and lower limits of the range of the coefficient $\tilde{C}$, respectively. Hence it follows that if some estimate of the temperature field $R^0(\tau, x)$ is known, a value of ρ can be selected for which the linear equation

$$R_\tau - \rho R_{xx} = -(\rho\tilde{C}(R^0) - 1)\,R_\tau^0 \tag{2.13}$$

roughly approximates initial nonlinear equation (2.8) in the sense that the solution of (2.13) with boundary conditions (2.9)-(2.11) is closer to the solution of problem (2.8)-(2.11) than R^0.

The following independent variables are introduced: $\bar{\tau} = \tau/(\rho b^2)$ and $\bar{x} = x/b$. Subsequently, the old notation τ and x will be used, implying $\bar{\tau}$ and $\bar{x}$. To solve the nonlinear IHCP the following iteration process is used:

1. The inverse problem

$$R_\tau = R_{xx},$$

$$R(0, x) = \tilde{U}_0^{l+1}(x), \quad R(\tau, 0) = \tilde{U}_1^{l+1}(\tau), \tag{2.14}$$

$$R_x(\tau, 1) = \tilde{g}^*(\tau), \quad R(\tau, x_k) = R_k^*(\tau) - R_1^l(\tau, x_k), \quad k = \overline{1, N}, \quad \tau \in (0, d_t).$$

The functions $\tilde{U}_0^{l+1}(x)$ and $\tilde{U}_1^{l+1}(\tau)$ are unknown. They should be found from the observations $R_k^*(\tau)$.

2. The direct problem (nonlinear):

$$R_\tau^{l+1} - R_{xx}^{l+1} = -(\rho\tilde{C}(R^{l+1}) - 1)\,R_\tau^{l+1}, \tag{2.15}$$

$$R^{l+1}(0, x) = \tilde{U}_0^{l+1}(x), \quad R^{l+1}(\tau, 0) = \tilde{U}_1^{l+1}(\tau), \quad R_x^{l+1}(\tau, 1) = \tilde{g}^*(\tau).$$

3. The direct problem (linear)

$$R_{1\tau}^{l+1} - R_{1xx}^{l+1} = -(\rho\widetilde{C}(R^{l+1}) - 1)\,R_{\tau}^{l+1}, \tag{2.16}$$

$$R_1^{l+1}(0, x) = 0, \quad R_1^{l+1}(\tau, 0) = 0, \quad R_{1x}^{l+1} = 0.$$

Here $\tau \in (0, d_t)$, $R_1^0(\tau, x) = 0$.

The following operators are introduced: the linear operator A: $R_0^l(\tau, \bar{x}) = A\widetilde{U}^l$, which operates according to (2.14), and the nonlinear operator A_1: $R_1^l(\tau, \bar{x}) = A_1(\widetilde{U}^l)$, which operates according to (2.15) and (2.16). Then, in operator form iteration process (2.14)-(2.16) can be written as

$$A\widetilde{U}^{l+1} = R^* - A_1(\widetilde{U}^l). \tag{2.17}$$

Process (2.17) is a method of successive approximations in which the linear IHCP must be solved for each iteration. With this approach, it is possible to relinquish the gradient methods that require solution of adjoint equations for calculation of the functional gradient and to simplify substantially the algorithm of the solution. In the finite-dimensional formulation, in the current iteration, Eq. (2.17) can be written in the form of the system of linear algebraic equations

$$A\bar{p} - \bar{z} = 0, \tag{2.18}$$

where $\bar{p} = [\bar{\beta}, \bar{\gamma}]^T$, $z_{i\times k} = R_k^*(\tau_i) - A_1(\bar{p}^l)(x_k, \tau_i)$, $i = \overline{1, m}$, $k = \overline{1, N}$, $\bar{p}$ is the vector of the sought coefficients, and $\bar{p}^l$ is the current approximation of $\bar{p}$. The solution of (2.18) is written in the form of (1.9).

It can be shown that the optimum value of the parameter ρ that specifies the linear model that approximates the initial nonlinear one in the best way in the sense of the minimum of $\|A_1(\bar{p})(x_k, \tau_i)\|_{R^{mN}}$ can be calculated from the formula

$$\rho^* = (A_2^T(\bar{p})\,\bar{y}(\bar{p}))/(\bar{y}^T(\bar{p})\,\bar{y}(\bar{p})), \tag{2.19}$$

where $\bar{y} = (A_1(\bar{p}) + A_2(\bar{p}))/\rho$; ρ is the value of the parameter that is used in calculation of $A_1(\bar{p})$ according to (2.15)-(2.16), and $A_2(\bar{p}) = R(x_k, \tau_i)$, where R is the solution of the problem

$$R_{\tau} - R_{xx} = R_{\tau}^{l+1}, \tag{2.20}$$

$$R(0, x) = 0, \quad R(\tau, 0) = 0, \quad R_x(\tau, 1) = 0.$$

Determination of ρ^* is, in essence, an iteration process, since as the value of the parameter ρ changes, the estimation interval changes in real time, which requires reformation of the observation sample used in solving inverse problem (2.14). However, practical calculations show that in most cases the first approximation obtained from results of solving the problem in the previous interval is sufficient, especially if the step d_s (see Fig. 1) is not too large. Since in this case, $A_1(\bar{p})$ is already known, it is only necessary to calculate additionally the vector $A_2(\bar{p})$ according to (2.20).

We consider the choice of the regularization parameter in solving linear IHCP (2.14) in the current iteration. With account for (1.9), the vector $\bar{p}$ is estimated from the formula

$$\hat{\bar{p}} = B\bar{z} = B(\bar{R}^* - A_1(\bar{p}^l)) = B(A\bar{p} + A_1'(\theta)(\bar{p} - \bar{p}^l) + \zeta).$$

With account for the last relation, the estimation error is specified by the expression

$$\bar{\varepsilon}(\alpha) = \bar{p} - \hat{\bar{p}} = (I - BA)\bar{p} + B\zeta + BA_1'(\theta)(\bar{p} - \bar{p}^l),$$

and the accuracy of the regularized solution, by the trace of the quadratic form $\Delta_N^2 = \mathrm{sp}(\mathrm{M}[\bar{\varepsilon}(\alpha)\bar{\varepsilon}^T(\alpha)])$. The iteration process converges when $\|B(\alpha)A_1'(\theta)\|_2 = k_0 < 1$.

Satisfiability of the last relation can be proved strictly only in the case of weak nonlinearity and with some limitations on the spectrum of the input signal; however, in most practical situations it is satisfied with a correct choice of ρ. In this case it can be demonstrated that

$$\lim_{i\to\infty} \Delta_N^2 = \Delta_L^2/(1 - k_0^2),$$

where Δ_L^2 is the accuracy of the solution of linear inverse problem (1.1).

Let $\bar{e}(\alpha) = (I - AB(\alpha))\bar{f}^* = E(\alpha)\bar{f}^*$ be the residual of problem (2.18), V_e be the covariance matrix of the residual, $\bar{\varepsilon}^l = \bar{p} - \bar{p}^l$ be the error of the current approximation, and $V_{\bar{\varepsilon}^l}$ be its covariance matrix. It can be shown that the solution that is optimum in the sense of the minimum of Δ_L^2 occurs in the case

$$V_{\bar{e}(B(\alpha^*))} = (V^* + V_\zeta)\, E^T, \tag{2.21}$$

where $V^* = A_1'(\theta)V_{\bar{\varepsilon}^l}(A_1'(\theta))^T$ and V_ζ is the transformed covariance matrix of the measurement error. The last expression means that the residual of problem (2.18) should be consistent with both the measurement error and the nonlinear contribution of the initial-approximation error. Therefore, for example, use of the heuristic rule

$$V_{\bar{e}(B(\alpha^*))} = f(l)\, V_\zeta\, E^T, \tag{2.22}$$

gives good results. Here $f(l)$ is a decreasing function of the iteration number l, equal to unity for numbers larger than the specified value. The last expression is considered to be a statistical hypothesis that can be checked using a statistic of the type of (1.12) and confidence interval (1.14).

We consider the expression for the residual of the nonlinear problem

$$\bar{e}^{\,l} = \bar{R}^* - A\bar{p}^{\,l} - A_1(\bar{p}^{\,l}) = A\bar{\varepsilon}^{\,l} + A_1'(\theta)\,\bar{\varepsilon}^{\,l} + \zeta.$$

In the case of the exact solution, $\bar{\varepsilon}^l = 0$, and consequently, $\bar{e}^l = \zeta$. Here the covariance matrix of the residual is equal to

$$V_{\bar{e}^l} = V_\zeta. \tag{2.23}$$

To check the statistical hypothesis W_1 about validity of equality (2.23), we use the statistic

$$r_{w_1} = (\bar{e}^{\,l})^T (V_\zeta)^{-1}\, \bar{e}^{\,l}. \tag{2.24}$$

If statistic (2.24) belongs to confidence interval (1.14), the hypothesis W_1 is correct and in this case iteration process (2.17) can be stopped.

Above we considered the inverse problem written in dimensionless form using the Fourier substitution. Let t_c be the current time. In dimensionless time the estimation interval is equal to d_t. Consequently, the real time interval and the step between measurements should be $\tilde{d}_t = \rho^* d_t/b^2$ and $\tilde{\delta}_t = \tilde{d}_t/m$. Measurements with a fixed time step are used most often in practice. This step is not identical to $\tilde{\delta}_t$, since the latter changes depending on the chosen ρ^*; however, a sequence of measurements with the step $\tilde{\delta}_t$ can be formed using interpolation. To do this, at the current time it is necessary to store measurements made in a previous time interval that is not shorter than $\tilde{d}_t^{min} = d_t(\tilde{C}_0 b^2)$, and the interval between measurements should not exceed $\tilde{\delta}_t^{max} = d_t/(\tilde{C}_1 b^2 m.)$

A series of computational experiments was carried out to investigate the suggested algorithm for solving the IHCP. Results of the solving the direct heat-conduction problem for some functional form of the boundary condition were used as exact initial data for solving the IHCP. The measurement error was simulated as white noise with a variance specified in percent of the maximum observed temperature $D = (1/3\tilde{\sigma}\max(f_k^*(\tau)))^2$. The

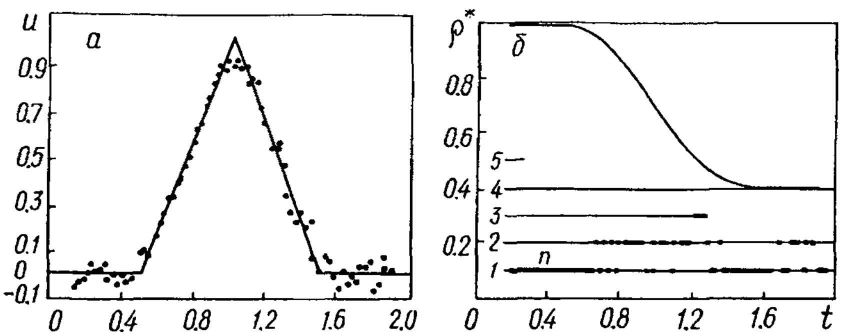

Fig. 5. Results of recovery of the heat flux: a) test example (solid line) and point estimates; b) optimum values of the parameter ρ^* and number of iterations; $N = 1$, $x_l = 1.0$, $d_t = 0.32$, $d_e = 0.16$, $m = 64$.

solution of the inverse problem was formed in the entire time interval as a simple sequence of point estimates. It was assumed that one temperature sensor located on the thermally isolated surface $x = b$ was used. The following parameters were used in the solution: $d_t = 0.32$; $d_e = 0.18$; $m = 64$; $\tilde{\sigma} = 5\%$, $C \equiv 1$, and $\lambda(T) = 1 + 3T$. In Fig. 5a one can see results of recovery of the heat flux, and Fig. 5b is a plot of the optimum value of the parameter ρ^* versus time. It also shows the number of iterations performed in each time interval. It can be seen that solution of the problem requires on average about two iterations. This confirms the rather rapid convergence of the method of successive approximations. Meanwhile, implementation of this method is much simpler and cost-saving in comparison with the gradient methods.

NOTATION

A, linear operator; A_l, operator of the nonlinear contribution; b, thickness of the plate; $B(\alpha)$, regularized inverse operator; $C(T)$, specific heat; $\tilde{C}(R) \in (\tilde{C}_0, \tilde{C}_1)$, reduced thermal diffusivity; d_t, magnitude of the estimation interval; d_e, delay of the estimate relative to current time; d_s, shift of the estimation interval; $\bar{e}(\alpha)$, residual vector; $E(\alpha)$, residual operator; $f_k^*(\tau)$, observed temperature values; $F_{[n\times n]}$, stabilization matrix; $g^*(\tau)$, $\tilde{g}^*(\tau)$, known boundary condition; $\bar{g}$, $\bar{g}_1$, intermediate result in calculation of the sought vector; $H_1(\omega)$, $H_2(\omega)$, transfer functions; i, index of the time node; k, number of the sensor; k_0, index of the convergence rate in the successive approximations; $K_1(\alpha)$, $K_2(\alpha)$, amplification factors; l, number of the iteration in the successive approximations; l_0, number of the iterations in choosing the regularization parameter; m, number of readings in the estimation interval; N, number of sensors; $\bar{p}_{[n]}$, vector of unknown coefficients; $\hat{\bar{p}}_{[n]}$, estimate of the vector $\bar{p}$; $r_w(\alpha)$, $r_{w_1}(\alpha)$, statistics; R, potential; $R_k^*(\tau)$, observed values of the potential; s_i, singular numbers; S, diagonal matrix of singular numbers; S_α, diagonal matrix that is inverse to the matrix S with account of $\alpha \neq 0$; t, time in the moving coordinate system; t_c, current time; t_e, time delay relative to t_c by d_e; T, temperature; $U_0(x)$, $\tilde{U}_0(x)$, initial condition for the temperature and potential; $U_1(\tau)$, $\tilde{U}_1(\tau)$, sought boundary condition for the temperature and potential; V_e, covariance matrix of the residual; V_ξ, V_ζ, covariance matrices of the noise of the measured temperature and potential; x, coordinate; x_k, coordinates of the location of the temperature sensors; $\bar{z}_{[mN]}$, observation vector; $\xi_k(\tau)$, $\zeta_k(\tau)$, noise of the measured temperature and potential; α, regularization parameter; β, parameter of linear transformation of the temperature; $\bar{\beta}_{[n1]}$, vector of approximation coefficients of the initial condition; γ, level of the first-order error; $\bar{\gamma}_{[n2]}$, vector of approximation coefficients of the boundary condition; Δ_L, Δ_N, accuracy of the solution of the linear and nonlinear inverse problems, respectively; $\bar{\varepsilon}(\alpha)$, estimation error of the coefficient vector; $\varepsilon(\alpha)$, $\tilde{\varepsilon}(\alpha)$, absolute and relative estimation errors of the boundary condition at time t_e; $\varphi_j(x)$, basis function; $\lambda(T)$, thermal conductivity; σ^2, variance of the noise of the measurements; $\tilde{\sigma}$, relative measurement error, %; ρ, parameter specifying the approximating linear model; ω, relative frequency; ω_0, frequency of the readings; τ, time in the moving coordinate system; Ξ_ε, guaranteed-estimation interval; Ξ_Θ, confidence interval of variation of the regularization parameter; $\psi_j(\tau)$, basis function; A^{-1}, A^T, inverse and transposed matrices; $\bar{p}_{[n]}$, vector $\bar{p}$ of dimension n; T_x, T_{xx}, first and second derivatives with respect to x; $\|*\|_{R^m}$, Euclidean norm of a vector of dimension m; $\|*\|_{W_2^2}$, norm of a function in the Sobolev space W_2^2; $\langle *,* \rangle_{W_2^2}$, scalar product in the Sobolevs space W_2^2; $\|*\|_2$, norm of a matrix, equal to the maximum singular number.

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